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A STRAIN GRADIENT THEORY OF PLASTICITY

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Abstract—A theory which includes first and second strain gradients is proposed as a model for plastic deformations. Heuristic arguments for including the gradients are also given. The main goal is to develop a logical framework in which behavior on a small scale can interact with the response on a larger one. The gradients lead to deformations which contain oscillatory components in addition to the ones obtained without them. For a particular material, these oscillatory deformations develop only during plastic deformations. They contain many of the properties that one would expect of the continuum analogue of dislocations. As contrasted with the theory of the continuous distribution of dislocations, the proposed theory applies to the entire plastic deformation process.

1. INTRODUCTION

IT IS very easy to observe plastic substances by performing an elementary tension test in which the material is loaded above some yield stress and the load removed (unloading). If permanent strains are found to occur when the load is completely removed, the material is said to be *inelastic*. If the stress–strain curve is independent of the time required for completing the cycle, the material is said to be *plastic*. Such materials are to be contrasted with viscoelastic or *visco-plastic* substances where the response functions involve the *rate of deformation*. Clearly a particular real material may be approximated as a plastic substance in some circumstances, as an elastic one in others and as visco-plastic in a still different class of problems. *Idealized materials* have no such variation in their responses, they are defined by constitutive relations and therefore *retain their properties* for all types of problems. For our present purposes we choose to consider materials whose response functions are independent of deformation rates; because this is the basic characteristic of materials which undergo plastic deformations. For convenience *only*, we also limit to small strains because the difference between this type problem and one with finite deformation measures is not critical.

The mathematical theory of plasticity that has become classical (see Hill [1]) is a complete analytical theory and is furthermore applicable to real materials under conditions which do not vary too much from those used to experimentally evaluate the response function. The difficulty is that real materials exhibit a wide variety of behaviors; and there does not appear to be a unified method for representing the response functions. For example the concept of "isotropic work hardening" applies very well to some materials but not at all to others. By considering nature in more detail, it is hoped that one could achieve a greater unity to the underlying structure of the response functions. It may not turn out to be the case because of the many different atomic and microscopic mechanisms already observed by physicists and metallurgists. However this remains a major goal of some researchers considering the foundations of plasticity.

A review of the present state of knowledge concerning plasticity will convince almost anyone that an improvement over the classical approach will come from considering small

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scale defects, especially "dislocations" in some way. The open question is, how? For some time it was believed that the continuous distribution of dislocations was an example of how to proceed from the microscopic scale to the normal continuum level. This procedure has recently [2] been said by some of its earlier proponents to be ineffective. Additional study of the present state of knowledge will show that a major need in continuum plasticity is an adequate representation of the "hardening" process and that physically this involves the *interaction* of dislocations with one another or with other defects. The "how" to incorporate this in a continuum theory is understood even less than how to consider dislocations themselves. For the purpose of having dislocations affect the response functions, the authors have recently [3] introduced an "internal" state variable approach. However the important question of "what is the continuum analogue of the dislocation" is not resolved by this procedure? Many people have many different ideas on what dislocations really are, but we believe it is fair to say there is no widely accepted analytical statement which defines this physical phenomena in a continuum manner.

Without being very precise, it is clear that dislocations occur at only a small percentage of atom sites in a chunk of a solid. When these dislocations interact to provide the hardening effect, they do so by forces whose range is *many atomic radii*. Now the classical concept of stress arises on the atomic scale as nearest neighbor interactions. Even in the calculations describing elastic materials, one considers (Lax [4]) that it is appropriate to introduce *higher spatial gradients* of the displacement in a continuum theory which is supposed to correspond to atomic level phenomena where the interactions are not of the nearest neighbor (i.e. not contact forces) type. Therefore we extend this concept of non local interactions to the case of dislocations by using higher gradients of the displacement as kinematical quantities entering the constitutive equations. Clearly the appropriate spatial *scale* is the *distance between dislocations*. We have not yet been able to justify the correspondence between displacement gradients and dislocation interaction by considering the energy distribution at the atomic level. However the analogy with elastic response provides sufficient motivation for the study.

Toupin [5] introduced the basic principles to be followed in continuum theories involving higher gradients when he assumed the energy–density to depend on the rotation gradients in addition to strain, and obtained the couple–stress theory of elasticity. In [5] he wondered why the strain-gradients did not enter his formulation and proceeded to show that this was basically due to a preconceived notion of the expression for work being done at a point on the surface of a body. Toupin and Gazis [6] then exhibited the correspondence between the first strain-gradient theory and the atomic lattice with nearest and next nearest neighbor interactions. Mindlin [7, 8] formulated the particular case of the second gradient of strain and related some terms in that theory to surface tensions in solids. Green and Rivlin [9] established a generalization of Toupin's basic work in which spatial gradients of the displacement of arbitrary order could be considered as well as independent "directors" of any order in materials designated as "multi-polar" media. Fox [10] recently gave an explicit correspondence between directors and dislocation parameters.

Our approach

We propose here a theory which includes higher spatial gradients of the displacements as a continuum level analogue of the dislocations and their interaction. The motivation for including higher gradients are: (1) the basic concept of work hardening due to Seeger [11] in which dislocations interact and therefore the internal forces are not restricted to being of the contact type; (2) the experimentalist using a better "strain gauge" (i.e. electron microscope or X-ray machine) observes a change in the *non-homogeneous* residual deformations in a standard tensile test if he also observes large scale plastic strains; (3) in hindsight the results of considering higher gradients contain such a nonhomogeneous strain field which changes with the deformation. This seems to us to be the major characteristic of dislocations which ought to be retained in going to a larger scale. In contrast to the theory of the continuous distribution of dislocations, we are able to study the entire process of plastic deformations (for the material with a quadratic free energy) and not just an after the fact calculation.

The calculations which are to follow are necessarily so complicated, that we *outline* here what we shall later do in detail.

1. *Second and third displacement gradients* are to be involved, so we must define the corresponding generalized stresses and the expression for the rate of doing work at a generic point on the surface of the body. These stresses may very well be zero on the surface of the body, but when we "localize" the conservation laws by saying that they apply to *every* element of the body, the higher gradients appear to be essential in the continuum model of nature in order to reflect the interaction of dislocations or other defects whose force fields act over many atomic distances. The proposed theory is considered to be a continuum theory but at the same time it provides knowledge of the deformation field over a smaller scale than we use in classical plasticity. In short it requires a better strain gauge to make the analogous experimental measurements, because there is a fine structure in the deformation field.

2. More detailed discussion of the *constitutive equations* is given. We find it convenient in our analysis to use a deformation field U_i which is the residual displacements in the body if the classical stresses are removed† from the point. We decided to introduce the field U_i, because in perfect plasticity it is the *continuum analogue* of the average displacement induced by dislocation motions. In perfect plasticity this leads to the "plastic work" being transformed completely into heat, which is really implied in atomic level calculations; it is of course the main feature of plasticity. It is the presence of U_i (in addition to the displacement function) which is the difference between our work and Mindlin's [7] elasticity theory.

3. The second law of thermodynamics is used in a standard Coleman–Noll axiomatic way for most of the variables and slightly novel way for the "plastic strain". That is the second law and other postulates are permitted to restrict the *domain* of the class of admissible processes as well as constitutive assumptions.

4. The theory is then illustrated for perfect plasticity and for a material in which the free energy is a quadratic function of the relevant variables. It turns out that perfect plasticity is simple and straight forward. Work hardening is then considered and requires a very much more complex framework. The basis of the difficulty in work hardening substances is that energy is "stored" at the atomic level in changing *nonhomogeneous* residual deformations, and this mechanism is significant for subsequent response of the material. We are not at all certain that we have the precise mechanism for energy storage. The real storage

† If the external forces are acting on the body, one imagines them removed in a quasi-static manner so that unloading inertia effects are *not* introduced. The displacement U_i is the vector from the initial position X_i of a generic point to the place it would occupy under these conditions. Alternatively, one can imagine a second body deforming without developing any classical stress (but with dipole and quadrapole stresses) in such a way as to reproduce the residual deformation of the real body.

mechanism may involve some variations of our formulation. In particular we shall use three spatial derivatives of the displacement whereas higher ones may be required for an accurate representation of nature.

We believe that this analytical model illustrates an acceptable framework for plasticity. It permits us to describe the *entire* process of plastic deformation in what we regard as a logically consistent manner and *this is our main goal!*

5. Except for the basic equations we shall use the small strain–small rotation concept and therefore for *convenience* use linear kinematic relations. These are not severe restrictions and are used for clarity of presentation.

2. BALANCE EQUATIONS

Toupin [5] very carefully expresses the balance laws for a material where the strain and gradients of rotation are considered. Except for the linear momentum, Green and Rivlin [9] bury the difficulties of the constitutive relations for higher momenta in pseudo-body force terms of appropriate tensor rank. Ralston and Jaunzemis [12] clearly illustrate the differences in the accelerations for strain-gradient theories as compared to atomistic terms. In different words they illustrate the fact that the constitutive equations for momenta in fancy materials are uncertain. We neglect here all momenta except the usual linear one.

We consider a generic particle originally at X_i to move to a point $x_i(X_\alpha, t)$ at time t. We utilize the Lagrangian approach and a rectangular cartesian coordinate system for convenience.

When the third spatial gradients of the deformation are sufficient, the energy balance equation is† assumed to be

$$\dot{E}^* + \dot{K} = \dot{W} - \int\int_A q_i n_i \,\mathrm{d}A + \int_{V_0} \rho_0 r \,\mathrm{d}v + \int_{V_0} f_i v_i \,\mathrm{d}v \tag{1}$$

where E^*, K and $\dot{W}$ are the internal energy, kinetic energy and rate of doing work by surface forces, respectively. The integrals in (1) are the contributions of the heat flow, energy source and the work of the body forces f_i, respectively. The integrals are over the initial volume and ρ_0 is the initial mass density. The rate of doing work $\dot{W}$ by the surface forces‡ is assumed to be

$$\dot{W} = \int\int_A (t_{ij} v_j n_i + t_{ijk} v_{i,j} n_k + t_{ijkl} v_{i,jk} n_l) \,\mathrm{d}A \tag{2}$$

where t_{ij}, t_{ijk} and t_{ijkl} are the stress, dipole and quadrapole stresses§ respectively. Clearly some of the t_{ijk} are "couple stresses" in the sense that this term has come to be used in recent years. In (2), v_i is the component of the velocity vector and n_i is the component of the normal to the surface. Considering a complete body the terms t_{ijk} and t_{ijkl} may very well vanish on the surface, but when we "localize" we need‖ these higher gradients if we consider applying the theory at the electron microscope scale (for example).

† We neglect writing body forces of all tensor ranks above the first. We use cartesian tensors and the summation convention for repeated indices. Superposed dots indicate material time derivatives.

‡ We use commas to denote partial derivatives, thus $v_{i,j} \equiv \partial v_i/\partial x_j$.

§ One can relate these to generalized tractions by including n_j (see Green and Rivlin [9] or Mindlin [7]).

‖ For a discussion of long range forces see Kröner [13]. This need does not arise from the divergence theorem, but rather from the postulate that localization is possible in the continuum model when non-nearest neighbors interact in the corresponding dislocation model.

Applying the divergence theorem to (2) one obtains

$$\dot{W} = \dot{W}^* + \dot{W}^{**}$$

where

$$\dot{W}^* = \int (t_{ij}v_{i,j} + t_{ijk}v_{i,jk} + t_{ijkl}v_{i,jkl})\, \mathrm{d}V \tag{3}$$

and

$$\dot{W}^{**} = \int (t_{ij,j} - t_{ijk,kj} + t_{ijkl,lkj})v_i\, \mathrm{d}V \tag{4}$$

$$+ \iint_A (t_{ijk,k}v_i + t_{ijkl,l}v_{i,k} - t_{ijkl,lk}v_i)n_j\, \mathrm{d}A.$$

We also require the second law of thermodynamics in the form of the Clausius–Duhem inequality

$$\rho_0\dot{S} - \frac{\rho_0 r}{T} + \left(\frac{q_\alpha}{T}\right)_{,\alpha} \geq 0 \tag{5}$$

where S is the entropy density, T is the absolute temperature.

3. ASSUMPTIONS

This section defines what we shall mean by a plastic material by listing a sequence of assumptions.

Assumption No. 1

The *initial* state is a configuration of material points in which the stress, dipole and quadrapole stresses all vanish, as does the heat flux vector. The temperature in this state is uniform at T_0 and the material is at rest in an inertial reference frame. We identify the material coordinates X_α with a material point X_α in this initial state.

By the motion of the material we shall mean† the continuous mappings $x_i = x_i(X_\alpha, t)$ giving the position x_i of each material point X_α for each instant of time t. The displacement of the generic particle is designated as u_i, hence $u_i = x_i - X_i$. The symmetric part of the first spatial gradient of u_i is the small strain tensor ε_{ij} whose components are

$$\varepsilon_{ij} = u_{(i,j)} \tag{6}$$

while the anti-symmetric part

$$\omega_{ij} = u_{[i,j]} \tag{7}$$

is the small rotation.

† It is clear that the material is being permitted to deform such that after some motion, the state where the stress, dipole and quadrapole stresses all vanish could not be mapped smoothly from the initial state. This difference in the unstressed states existing at $t = 0$ and at $t = t$, is normally the basis for introducing dislocations in the material. We prefer to retain the continuous mappings and to permit changing residual stress fields (especially dipole and quadrapole stresses) as part of the motion.

We associate the name *second displacement gradients* α_{ijk} with

$$\alpha_{ijk} = u_{i,jk} \tag{8}$$

and α_{ijkl} with the *third spatial gradient* of u_i

$$\alpha_{ijkl} = u_{i,jkl}. \tag{9}$$

Clearly there are certain symmetries implied in the relations (8) and (9). There is some advantage in separating them into symmetric and anti-symmetric terms but we do not do this in order to maintain a conciseness in notation.

Assumption No. 2

As in classical continuum mechanics we assume the momentum/unit mass is equal to the velocity $\dot{x}_i$. This means that we are going to neglect gradient (dislocation) dynamics. Then the kinetic energy is assumed to be

$$2K = \int_{V_0} \rho_0 \dot{x}_i \dot{x}_i \, \mathrm{d}V. \tag{10}$$

Assumption No. 3

We now assume that internal energy rate $\dot{E}^*$ *is the same* when expressed in two co-ordinate systems having a constant relative velocity. When this is true, (1) and the definition of K imply

$$\int (t_{ij,j} + t_{ijk,kj} + t_{ijkl,lkj} + f_i - \rho_0 \dot{v}_i) v_i \, \mathrm{d}V = 0$$

which is the linear momentum equation for a material with higher stresses. Hence if we assert that the balance laws must apply to every part of the body, we obtain

$$t_{ij,j} + t_{ijk,kj} + t_{ijkl,lkj} + f_i = \rho_0 \dot{v}_i \tag{11}$$

Assumption No. 3(a)

We now assume that the internal energy rate $\dot{E}^*$ *is the same* when expressed in two co-ordinate systems having a constant relative angular velocity. When this is true; the above integral, (1) and the definition of K imply

$$\int (t_{[ij]} \dot{\omega}_{ij} \, \mathrm{d}V + \int \int_A t_{[ik]jl,l} \dot{\omega}_{ik} n_j \, \mathrm{d}A = 0.$$

Hence the differential form of the angular momentum equation

$$t_{[ik]} + t_{[ik]jl,lj} = 0$$

and

$$t_{[ik]jl,l} \dot{\omega}_{ik,j} = 0$$

is obtained with the aid of the divergence theorem. In the present theory these equations will turn out to be identically zero and the stress tensor symmetric [due to constitutive equation (13)]. These equations are to be subjected to boundary conditions where the integrands of the surface integral in (4) is specified at each point on the boundary just as in second strain gradient elasticity [7].

It is in the transition from the integral to the differential form (11) that we introduce the need for higher spatial variations in u_i. That is, when we assert that the integral applies to every element of the body, we must introduce higher derivatives to obtain differential equations representing non-local action at a point. The energy equation in differential form now becomes

$$\dot{E} = t_{ij}v_{i,j} + t_{ijk}v_{i,jk} + t_{ijkl}v_{i,jkl} - q_{i,i} + \rho_0 r. \tag{11a}$$

Assumption No. 4

We introduce a pseudo-motion $y_i(X_\alpha, t)$ of a material point, which is the mapping y_i which gives the position y_i that each material point X_α *would take at time t*, if all the classical stresses t_{ij} were removed from the body quasi-statically; i.e. so that unloading inertia effects would not be introduced. For example, one imagines this is done by a suitable body force or temperature distributions. Approximately, y_i is the mapping from the initial state to the final actual state in a test in which permanent homogeneous deformations are produced. We associate the name residual displacement U_i with the change in position from the initial state to the position y_i, so that

$$U_i \equiv y_i - X_i.$$

We associate the name *residual strain* P_{ij} with the symmetric part of the first gradient of U_i

$$P_{ij} = U_{(i,j)}. \tag{12}$$

We next introduce the gradients of the residual displacements U_i as the gradients of the deformation u_i plus a perturbation term β_{ijk} and β_{ijkl} defined by

$$\frac{\partial^2 U_i}{\partial x_j \partial x_k} = \alpha_{ijk} + \beta_{ijk}, \tag{12a}$$

$$\frac{\partial^3 U_i}{\partial x_j \partial x_k \partial x_l} = \alpha_{ijkl} + \beta_{ijkl}. \tag{12b}$$

Assumption No. 5

We here consider that the *entire process* of plastic deformations is to be included† in a single analysis. Since plastic materials can sometimes be elastic, we include all variables associated with thermoelasticity without much other justification. Most of the energy associated with plastic deformation is dissipated as heat (Taylor [15], Dillon [16]), hence P_{ij} will be‡ included as an intermediate independent variable in constitutive equations. Other "physical" evidence (Kroupa [17], Seeger [11], Kröner [18]) suggests that the part of the energy that is "stored" is primarily in the nature of "tangling" which is here considered to be synonymous with the value of α_{ijkl}. Energy is also stored as the number of dislocations increases. We consider that the corresponding phenomena in a continuum theory is a change in energy as α_{ijk} is altered.

† The study of dislocation behavior in strain-rate independent materials usually does not involve the generation of the dislocations and the changing structure of their arrangement, on the other hand work hardening theories require this [11]. The generation of dislocations require new assumptions, such as Frank–Read sources.

‡ Since P_{ij} is not a state variable, there is some reason not to use it explicitly in constitutive equations. It is our view here that one can use P_{ij}.

To be explicit we assume† that the dependent variables $(t_{ij}, t_{ijk}, t_{ijkl}, q_k, \phi, S)$ all depend on the current values of $(\varepsilon_{ij}, P_{ij}, \alpha_{ijk}, \alpha_{ijkl}, T)$ and are independent of the history and rates of these parameters. For example the free energy ϕ is

$$\phi = \phi(\varepsilon_{ij}, P_{ij}, \alpha_{ijk}, \alpha_{ijkl}, T). \tag{13}$$

By not including the β_{ijk} and β_{ijkl} in the constitutive relations (13) we are in effect neglecting the elastic couple stresses. These stresses arise at the atomic level due to non nearest neighbor interactions which extend only over a few *atomic* radii. The gradient terms being retained extend over a *few dislocation separation distances* but this is many atomic distances.

Because plastic materials respond differently in hydrostatic solutions than in deviatoric deformations, we shall separate the stresses and strains, so that $\phi(\varepsilon_{ij}) = \phi(e_{ij}, \varepsilon_{kk})$ where

$$\varepsilon_{ij} = \frac{\varepsilon_{kk}}{3}\delta_{ij} + e_{ij} \tag{14}$$

$$e_{ii} = 0,$$

where ε_{kk} is the dilatation and e_{ij} are the components of the strain deviator. The stresses are similarly divided such that

$$S_{ij} = t_{ij} - t_{kk}\delta_{ij}/3 \tag{14a}$$

is the deviator of stress and t_{kk} is the hydrostatic tension. We shall consider $P_{ii} = 0$ because plastic deformations are incompressible.

We emphasize that the constitutive equations for all of the dependent variables are in the form (13) and that they are completely independent of the history and rates of deformation. This is in contrast to viscoelastic or viscoplastic materials.

Assumption No. 5(a)

The free energy is assumed to explicitly contain P_{ij}, i.e.

$$\frac{\partial\phi}{\partial P_{ij}} \neq 0. \tag{15}$$

It is the assumption which keeps the present analysis in plasticity and does not permit the unwanted return to elasticity. In other words (15) prevents P_{ij} from disappearing from ϕ in the same way that the temperature gradients frequently do in this type of analysis. P_{ij} is included in (13) to permit plastic work to go into heat. Thus it permits one to reasonably study the *process of developing plastic deformation* where small temperature changes are an important feature of the conservation of energy.

While the temperature changes are small, they are energetically equivalent to the plastic work, which is thought to be important (e.g. Hill [1], p. 26) in conventional plasticity theories which are used for work hardening materials.

† The analytical material to be examined herein is *precisely defined* by (13) and similar equations for $(t_{ij}, t_{ijk}, t_{ijkl}, q_i, S)$. The discussion on dislocations is intended only to motivate the list of intermediate variables included in (13). We believe that higher gradients can be loosely related to dislocations in one's mind. The *precise* connection however has not been established.

Assumption No. 6

There is a region in the neighborhood of zero classical stresses where the material is linearly elastic. That is $\dot{P}_{ij} \equiv 0$ for processes in some regions of the space spanned by the arguments listed in (13). It is convenient to introduce the boundary of these reversible domains by means of a yield function $f(S_{ij}, t_{ijk}, t_{ijkl}, T) = 0$ and by convention we take

$$f < 0 \quad \text{when } \dot{P}_{ij} \equiv 0 \tag{16a}$$

and

$$f \geq 0 \quad \text{when } \dot{P}_{ij} \neq 0. \tag{16b}$$

Real materials probably "yield" at very small stresses but the resulting permanent strains are difficult to measure. Clearly the set of processes which satisfy both the second law (5) and (16b) may be empty. This simply means the assumptions were made incorrectly and new ones must be used.

Remark 1

For materials whose response functions do not depend on time or rates of deformation, consistency indicates that the yield function should also have this property. Thus we *assume* that

$$\dot{f} = 0 \quad \text{when } \dot{P}_{ij} \neq 0 \tag{16c}$$

so that processes where $\dot{P}_{ij} \neq 0$ remain on the yield surface $f = 0$. Said differently when one considers processes where $\dot{P}_{ij} \neq 0$, subsequently unloads and thereafter reloads, plastic flow commences when $f = 0$. However this can be exactly the same state from which unloading occurred, hence f must also vanish just prior to the unloading. Hence the value of f is assumed† to be unchanged when plastic flow occurs.

An example of the type yield functions that seem to be appropriate to gradient theories is

$$f = f_{ij}\bar{f}_{ij} \tag{16d}$$

where

$$f_{ij} = S_{ij} - K_k t_{ijk} - K_{kl} t_{ijkl} - Y_{ij} \tag{16e}$$

$$\bar{f}_{ij} = S_{ij} - K_k t_{ijk} + K_{kl} t_{ijkl} + Y_{ij}. \tag{16f}$$

In (16e) K_k, K_{kl} and Y_{ij} are "constants" of the appropriate tensor rank. In effect the yield function (16d) represents the Von Mises criterion which expands in a manner related to the values of the localized stresses.

We shall also assume that $\dot{P}_{ij} = 0$ when

$$f = 0 \quad \text{but} \quad \dot{f} < 0 \tag{16g}$$

as is customary in plasticity.

† This assumption idealizes real material response. If f increases, one is in the viscoplastic region.

Assumption No. 7

The condition of invariance of the free energy density ϕ in a rigid rotation of the deformed body in the present case is taken as

$$\frac{\partial \phi}{\partial \omega_{ij}} = 0. \tag{17}$$

Assumption No. 8

The deviatoric stresses are related to the free energy by the stress relation.

$$S_{ij} - \rho_0 \frac{\partial \phi}{\partial e_{ij}} = 0. \tag{18}$$

For some materials we shall later see that this assumption is not needed but can be deduced from the others. We prefer to bring it explicitly into the open because it is not a major point in the paper. We would emphasize that it is applicable to loading and to unloading.

4. SECOND LAW

The main differences between Mindlin's second gradient elasticity [7] and plasticity theory proposed here are the inclusion of P_{ij} in (13) and the corresponding interpretation of the second law of thermodynamics. We find it convenient to introduce the free energy density ϕ by

$$E = \phi + TS \tag{19}$$

where $E \equiv E^*$ after the linear and angular momentum terms are deleted. Considering (11a) and (19) we have the energy equation in the form

$$\rho_0 \dot{\phi} = \rho_0 r - q_{\alpha,\alpha} - \rho_0 \dot{S} T - \rho_0 S \dot{T} + t_{ij} \dot{\varepsilon}_{ij} + t_{ijk} v_{i,jk} + t_{ijkl} v_{i,jkl} \tag{20}$$

and the second law (5) is

$$t_{ij} \dot{\varepsilon}_{ij} + t_{ijk} v_{i,jk} + t_{ijkl} v_{i,jkl} - \rho_0 \dot{\phi} - \rho_0 S \dot{T} - \frac{q_\alpha T_\alpha}{T} \geq 0. \tag{21}$$

Because of (8), (9) and (14), this becomes

$$S_{ij} \dot{e}_{ij} + t_{kk} \dot{\varepsilon}_{ll}/3 + t_{ijk} \dot{\alpha}_{ijk} + t_{ijkl} \dot{\alpha}_{ijkl} - \rho_0 \dot{\phi} - \rho_0 S \dot{T} - \frac{q_\alpha T_\alpha}{T} \geq 0 \tag{22}$$

where S_{ij} is the stress deviator defined by

$$t_{ij} = t_{kk} \delta_{ij}/3 + S_{ij}. \tag{23}$$

The set of functions $(S_{ij}, t_{kk}, \phi, S, q_\alpha, t_{ijk}, t_{ijkl})$ define a *process* if they satisfy the balance laws. They are an *admissible process* in plastic materials if they are (a) a process, and (b) satisfy the equations analogous to (13). Considering that the free energy, all the stresses, the heat flux and entropy satisfy equation (13), (21) becomes

$$\left(S_{ij} - \rho_0 \frac{\partial \phi}{\partial e_{ij}}\right) \dot{e}_{ij} + \left(\frac{t_{kk}}{3} - \rho_0 \frac{\partial \phi}{\partial \varepsilon_{kk}}\right) \dot{\varepsilon}_{ll} - \rho_0 \left(\frac{\partial \phi}{\partial T} + S\right) \dot{T} - \frac{q_\alpha T_\alpha}{T}$$
$$+ \left(t_{kij} - \rho_0 \frac{\partial \phi}{\partial \alpha_{ijk}}\right) \dot{\alpha}_{ijk} + \left(t_{kijl} - \rho_0 \frac{\partial \phi}{\partial \alpha_{kijl}}\right) \dot{\alpha}_{kijl} - \rho_0 \frac{\partial \phi}{\partial P_{ij}} \dot{P}_{ij} \geq 0. \tag{24}$$

We next consider certain *convenient admissible processes* where the balance equations are trivially satisfied, but the second law may not be. Since the *rates* in the second law are not fixed by the balance laws or the constitutive equations we choose them to suit our convenience, and are able to make certain *general* deductions of significance. This is the Coleman–Noll approach to thermodynamics of deformable media.

Remark 2

One of the major contributions of Coleman and Noll [19] was the clarity of the word process in their analysis. Thus we remark that the knowledge of u_i, U_i and T as functions of X_α and t (within the body and during the appropriate time interval) are *sufficient* to describe an admissible process. In point of fact U_i is related to u_i and T when the detailed process is being considered, but it is difficult to provide a general relationship. On the other hand the strain rates derived from u_i and U_i will be related through the condition that $\dot{f} = 0$ when $\dot{P}_{ij} \neq 0$. Clearly only u_i and T can be controlled independently in a process.

The proof of the above remark is: since u_i, U_i and T are known, then ε_{ij}, P_{ij}, α_{ijk}, α_{ijkl} are also known. Once these are evaluated, the quantities (t_{ij}, t_{ijk}, t_{ijkl}, q_k, ϕ, s) are in turn determined by (13). Once these are calculated the gradients in (11) and (20) are known, by choosing body forces f_i and heat supply r, one satisfies (11) and (20) identically.

In general one must also check the angular momentum equations as well and utilize higher types of body couples. These higher couples have not been carried along in order to simplify the presentation. In what follows the angular momentum equations are simply identities when (as shall be our case) the stresses t_{ijk} and t_{ijkl} are symmetric in the first two indices due to constitutive assumptions.

Now let u_i, T, U_i be an arbitrary point such that ε_{ij}, P_{ij}, etc. are within the domain of the response functions for (ϕ, S, t_{ij}, t_{ijk}, t_{ijkl}, q_k) for the material point X'_α. Choosing arbitrarily a time t_0, tensors $\bar{\varepsilon}_{ij}$, $\bar{P}_{ij}$, $\bar{\alpha}_{ijk}$, $\bar{\alpha}_{ijkl}$, A_{ij}, B_{ij}, A_{ijk}, B_{ijk}, B_{ijkl} as well as vectors g_i and c_i and the scalar a, we consider the displacements u_i, U_i and the temperature distributions defined by

$$\begin{aligned} u_i = {} & \bar{u}_i+[\bar{\varepsilon}_{ij}+(t-t_0)A_{ij}][X_j-X'_j] \\ & +[\bar{\alpha}_{ijk}+(t-t_0)A_{ijk}][X_j-X'_j][X_k-X'_k] \\ & +[\bar{\alpha}_{ijkl}+(t-t_0)A_{ijkl}][X_j-X'_j][X_k-X'_k][X_l-X'_l], \end{aligned} \tag{25}$$

$$\begin{aligned} U_i = {} & \bar{U}_i+[\bar{P}_{ij}+(t-t_0)B_{ij}][X_j-X'_j] \\ & +[\bar{\alpha}_{ijk}+\bar{\beta}_{ijk}+(t-t_0)B_{ijk}][X_j-X'_j][X_k-X'_k] \\ & +[\bar{\alpha}_{ijkl}+\bar{\beta}_{ijkl}+(t-t_0)B_{ijkl}][X_j-X'_j][X_k-X'_k][X_l-X'_l], \end{aligned} \tag{26}$$

$$T = \bar{T}(X')+(t-t_0)a+[\bar{g}_i+(t-t_0)c_1][X_j-X'_j], \tag{27}$$

for all points in the body and for all $t \geq t_0$ sufficiently close to t_0. Because of the large number of parameters included here, we shall automatically assume that the quantities to be listed are zero, unless explicitly declared to be different than zero. They are

$$\begin{aligned} & A_{ij} = A_{ijk} = A_{ijkl} = B_{ij} = B_{ijk} = B_{ijkl} = 0, \\ & a = c_i = 0. \end{aligned}$$

Convenient processes No. 1

The stresses and strains satisfy the balance laws identically but the temperature is imagined to be changed uniformly, such that the above quantities vanish except

$$a \neq 0.$$

In this case (24) reduces to

$$-\rho_0\left(\frac{\partial\phi}{\partial T}+S\right)a \geq 0. \tag{28}$$

Since a can be plus or minus, or indeed changed suddenly from positive to negative, we deduce that

$$\frac{\partial\phi}{\partial T}+S \equiv 0. \tag{29}$$

Since (29) is independent of rates by (13), this deletes the third term from (24) for all remaining processes.

Convenient process No. 2

We then consider a process which changes the volume uniformly, so that

$$A_{ll} \neq 0.$$

Hence

$$t_{kk}/3-\rho_0\partial\phi/\partial\varepsilon_{kk} = 0 \tag{30}$$

since by assumption No. 4, $P_{ii} = 0$.

Convenient process No. 3

We consider that it is possible to strain the material homogeneously in shear at uniform temperature, so that A_{ij} and B_{ij} do not vanish and hence (24) becomes

$$\left(S_{ij}-\rho_0\frac{\partial\phi}{\partial e_{ij}}\right)A_{ij}-\rho_0\frac{\partial\phi}{\partial P_{ij}}B_{ij} \geq 0. \tag{31}$$

If $f < 0$, then $B_{ij} = 0$ and we obtain

$$S_{ij}-\rho_0\frac{\partial\phi}{\partial e_{ij}} \tag{18'}$$

since A_{ij} is arbitrary. Furthermore if $f = 0$ and $\dot{f} < 0$, B_{ij} also vanishes. The coefficients of A_{ij} and B_{ij} in (31) do not contain "rates" and hence the same result must be obtained whether or not $\dot{f} = 0$ or $\dot{f} < 0$. Hence (18)′ is a result that must hold for all processes in the class of materials being considered. It is the same relationship as is obtained in elastic materials. Equation (31) then becomes

$$-\rho_0\left(\frac{\partial\phi}{\partial P_{ij}}\right)B_{ij} \geq 0. \tag{32}$$

The result (32) is really at the base of the explicit results to be developed below for it restricts the *domain* of admissible processes, just as the yield function condition $\dot{f} = 0$ does. It indicates that B_{ij} are not completely arbitrary (as A_{ij} are) but in fact are related to $\bar{\varepsilon}_{ij}$, $\bar{P}_{ij}$, $\bar{\alpha}_{ijkl}$, $\bar{\alpha}_{ijkl}$, etc. It will turn out that the A_{ij} and B_{ij} are also related by the condition $\dot{f} = 0$. Hence (32) and $\dot{f} = 0$ must be mutually consistent, which says that the yield condition and the free energy are related. The relationship is however quite easy to satisfy as indicated in the special examples given below. In permitting the second law to restrict *domains* of processes as well as constitutive assumptions we utilize the ideas of Leigh [22]. In fact, it seems to us that the second law really only verifies consistency of the other assumptions concerning constitutive relations and process dbmains.

Suppose that one insists that (32) implies restrictions only on the constitutive relations. Then (32) requires different expressions for "loading" and for "unloading". The end result (actual process) is however precisely the same in either case. The requirement for modifying ϕ is awkward and we prefer the restriction on the process domain as a matter of convenience. By allowing a restriction of the process domain, one can use single valued constitutive relations for plastic materials. If initially we have a process where $\dot{P}_{ij}$ is positive (and therefore $\partial\phi/\partial P_{ij}$ is negative) and we attempt to reverse $\dot{P}_{ij}$, we cannot do so and also satisfy (32). Hence (32) implies $\dot{P}_{ij} = 0$ under these conditions.

Convenient special process No. 4

We consider that the temperature can be changed nonuniformly without inducing plastic deformations or displacement gradients. Hence c_i does not vanish and we obtain the usual condition

$$-q_\alpha T_{,\alpha} \geq 0 \tag{33}$$

for the relationship between the heat flux vector and the temperature gradient under these conditions.

Convenient process No. 5

As noted in process No. 3, the inequality coefficients do not depend on rates and hence the same result must be obtained when $\dot{f} = 0$ and $\dot{f} < 0$. Hence, considering the results (28)–(31), (24) becomes

$$\left(t_{ijk} - \rho_0 \frac{\partial\phi}{\partial\alpha_{ijk}}\right) A_{ijk} + \left(t_{ijkl} - \rho_0 \frac{\partial\phi}{\partial\alpha_{ijkl}}\right) A_{ijkl} - \rho_0 \frac{\partial\phi}{\partial P_{ij}} \dot{P}_{ij} \geq 0$$

and implies that

$$t_{ijk} - \rho_0 \frac{\partial\phi}{\partial\alpha_{ijk}} = 0, \tag{34}$$

$$t_{ijkl} - \rho_0 \frac{\partial\phi}{\partial\alpha_{ijkl}} = 0. \tag{35}$$

The results so far obtained in this paper can be summarized in a theorem which give necessary and sufficient conditions for the second law (5) to hold.

Theorem 1

Consider any body B made of a plastic material such that the dependent variables $(S_{ij}, t_{kk}, S, t_{ijk}, t_{ijkl}, \phi, q_\alpha)$ are single valued functions of the intermediate variables $(e_{ij}, \varepsilon_{kk}, P_{ij}, \alpha_{ijk}, \alpha_{ijkl}, T)$ in such a way that (a) $\dot{P}_{ij} \neq 0$ on loading (b) $\partial\phi/\partial P_{ij} \neq 0$; (c) equation (11) is the momentum equation; (d) the deviatoric stress is given by $S_{ij} = \rho_0 \partial\phi/\partial e_{ij}$; let these conditions hold at each point X_α in (B) and the dependent variables be sufficiently continuous in their arguments. In order that the postulate (5) hold for all smooth admissible processes in (B), it is necessary and sufficient that the following statements be true for each point X_α in (B).

(i) The entropy and stresses are related to the free energy through

$$S + \partial\phi/\partial T = 0,$$

$$t_{kk} - 3\rho_0 \partial\phi/\partial\varepsilon_{kk} = 0,$$

$$t_{ijk} - \rho_0 \partial\phi/\partial\alpha_{ijk} = 0$$

and

$$t_{ijkl} - \rho_0 \partial\phi/\partial\alpha_{ijkl} = 0.$$

(ii) The heat flux vector, temperature gradient and residual strain obey the inequality

$$-\rho_0 \frac{\partial\phi}{\partial P_{ij}} \dot{P}_{ij} - q_\alpha T_{,\alpha} \geq 0$$

since in general (24) cannot be reduced to (32) and (33).

Energy equation

Under the conditions of the above theorem the energy equation (20) becomes

$$-q_{\alpha,\alpha} + \rho_0 r = \rho_0 T\dot{S} + \rho_0 \frac{\partial\phi}{\partial P_{i\alpha}} \dot{P}_{i\alpha} \tag{36}$$

where, of course, the last term is restricted by (32). For some purposes it is convenient to have the value of the "stored energy" χ which is simply the difference between the work done by all external forces and the heat releases in an adiabatic process. In the present context this is

$$\dot{\chi} = \dot{E} - \rho_0 C_D \dot{T} - \dot{W}_{\text{rev}} \tag{37}$$

where $\dot{W}_{\text{rev}}$ is the reversible part of the working of the generalized forces $t_{ij}n_j$, $t_{ijk}n_j$ and $t_{ijkl}n_j$. The stored energy has been very important to metallurgists and it is this term which one normally calculates from the continuous contribution of dislocations. The key point (to us) is that one cannot tell about the consistency of the stored energy unless it is part of a theory encompassing the plastic deformation *process* such as that presented here.

5. PERFECT PLASTICITY

According to common usage elastic–perfectly plastic materials, do not exhibit work hardening. Within the framework of the present theory, this means that there is no storage of energy in a process where the classical stresses produce plastic deformations and are subsequently removed. In view of the discussion in assumption No. 5, this suggests using a free energy which does not contain α_{ijk} or α_{ijkl}. We shall now consider a very special (limited application) case of such a material and examine what happens.

Consider a material whose yield function is

$$f = 2\mu(e_{ij}-P_{ij})(e_{ij}-P_{ij})-K^2 \tag{38}$$

and whose free energy is

$$\rho_0\phi = 2\mu(e_{ij}-P_{ij})(e_{ij}-P_{ij})/2+(3\lambda+2\mu)(\varepsilon_{kk}-\alpha[T-T_0])^2/2$$
$$-\rho_0 C_D(T-T_0)^2/2T_0+K_{ijkl}y_{kl}P_{ij}T/T_0. \tag{39}$$

where α, λ, μ, C_D are the normal coefficient of thermal expansion, Lame's constants and specific heat, respectively. The last term will be discussed below. For this material, the deviatoric stresses are

$$S_{ij} = 2\mu(e_{ij}-P_{ij}) \tag{40}$$

for all process and Theorem 1 yields

$$t_{kk} = (3\lambda+2\mu)[\varepsilon_{kk}-\alpha(T-T_0)]$$

and

$$-\rho_0\frac{\partial\phi}{\partial P_{ij}}\dot{P}_{ij} = \left[2\mu(e_{ij}-P_{ij})-K_{ijkl}y_{kl}\frac{T}{T_0}\right]\dot{P}_{ij} \geq 0 \tag{41}$$

hence we have, from Theorem 1 when $T_{,i} = 0$

$$(S_{ij}-K_{ijkl}y_{kl}T/T_0)\dot{P}_{ij} \geq 0 \tag{42}$$

and from $\dot{f} = 0$, $S_{ij}(\dot{e}_{ij}-\dot{P}_{ij}) = 0$ when $f = 0$. Clearly $K_{ijkl}y_{kl}$ can be interpreted as the locus of stress states where plastic flow commences. It is required to be non-zero by virtue of constitutive assumption No. 5. If we assume: (a) isotropic materials, (b) $y_{kl} = y_{lk}$, note that $P_{ii} = 0$ and that initially $T/T_0 = 1{\cdot}0$ (42) reduces to

$$(S_{ij}-Ky_{ij})P_{ij} \geq 0 \tag{42a}$$

where K is a constant and y_{ij} is the "shape factor" of the yield condition in stress space. It is discussed clearly in [1, p. 18] in terms of the usual yield condition. The Ky_{ij} are not changed by the deformation in perfect plasticity so that "unloading" and subsequent "reloading" leads to plastic flow commencing under the same conditions as in the initial situation.

The entropy is

$$\rho_0 S = +(3\lambda+2\mu)(\varepsilon_{kk}-\alpha(T-T_0))+\rho_0 C_D(T-T_0)/T_0-Ky_{ij}P_{ij}/T_0$$

and the energy equation (33) becomes

$$-q_{\alpha,\alpha}+\rho_0 r = \rho_0 C_D\dot{T}+\rho_0(3\lambda+2\mu)(\dot{\varepsilon}_{kk})-S_{ij}\dot{P}_{ij}. \tag{43}$$

Hence in an adiabatic shearing motion in which $r = q_\alpha = \dot{\varepsilon}_{kk} = 0$, all of the plastic work in a material defined by (39) appears as heat. The reversible work rate $\dot{W}_{\text{rev}} = S_{ij}\dot{S}_{ij}/2\mu$ in (37). Hence

$$\dot{\chi} = 0 \tag{44}$$

so that the absence of α_{ijk} and α_{ijkl} is synonymous with the rate of storing energy also being identically zero in this model.

From (40) and the condition $\dot{f} = 0$, one sees that the stress does not exceed the yield value in one dimension. In two dimensions the stress rate must be orthogonal to the present value.

Consider now the case of unloading from a yield surface process. Then (41) becomes

$$2\mu(\dot{e}_{ij}-\dot{P}_{ij})\Delta t\dot{P}_{ij} \geq 0$$

from which one concludes that when $\dot{e}_{ij}$ is negative the condition that $|\dot{e}_{ij}| > |\dot{P}_{ij}|$ and hence the stress falls below the yield value in which case assumption No. 6 is again operative and the response is elastic.

6. LINEAR WORK HARDENING

In order to obtain an illustrative example which includes work hardening, we consider a material whose ϕ is quadratic in the parameters. Many of the relations to be obtained here are formally similar to Mindlin's results [7] in linear elasticity with second strain gradients.

We assume a free energy such that

$$\begin{aligned}\rho_0 2\phi = {} & 2\mu(e_{ij}-P_{ij})(e_{ij}-P_{ij})+(3\lambda+2\mu)[\varepsilon_{kk}-\alpha(T-T_0)]\{\varepsilon_{ll}-\alpha(T-T_0)\}\\ & +a_{ijklmn}\alpha_{ijk}\alpha_{lmn}+b_{ijklmnpq}\alpha_{ijkl}\alpha_{mnpq}\\ & +2c_{ijklmn}\alpha_{klmn}P_{ij}+2Ky_{ij}P_{ij}T/T_0-\rho_0 C_D(T-T_0)^2/T_0.\end{aligned} \tag{45}$$

For centro-symmetric (isotropic) materials the a, b and c tensors reduce to combinations of Kronecker deltas (see Jaunzemis [20] p. 302 and Mindlin [7]). Since we consider our material to be elastic in dilatation our results are slightly simpler than the corresponding case in elasticity. In particular (38) and Theorem 1 imply

$$\begin{aligned}S_{pq} &= 2\mu(e_{pq}-P_{pq})\\ t_{pqr} &= 2a_4\alpha_{pqr}+a_5(\alpha_{rqp}+\alpha_{rpq})\end{aligned} \tag{46}$$

$$\begin{aligned}t_{pqrs} = {} & \frac{2b_2}{3}\alpha_{jkii}\delta_{jkpqrs}+\frac{b_3}{3}\alpha_{jsii}\delta_{jpqr}\\ & +2b_6\alpha_{pqrs}+\frac{2b_7}{3}(\alpha_{qrsp}+\alpha_{rspq}+\alpha_{spqr})\\ & +\frac{c_2P_{ij}}{3}\delta_{ijpqrs}+\frac{c_3}{3}P_{is}\delta_{pipr}\end{aligned} \tag{47}$$

where δ_{ipqr} and δ_{ijpqrs} are combinations of Kronecker delta symbols defined in [7]. The displacement equations of equilibrium are obtained from combinations of (11), (46) and (47) as well as (6) and (12). From these or from (47) one sees that the physical significance of the parameters a_4, a_5, b_2, b_3, b_6 and b_7 is to permit generalized couple stresses to develop in the *elastic* range. Since the β_{ijk} and β_{ijkl} defined by (12) are neglected in the constitutive equations (13), consistency suggests that we assume that all elastic couple stresses vanish and therefore that these parameters should vanish. Henceforth we shall consider that

$$a_4 = a_5 = b_2 = b_3 = b_6 = b_7 = 0. \tag{48}$$

Under the conditions (48), the displacement equations of equilibrium become

$$\nabla^2 u_i = \nabla^2 U_i - (\tfrac{1}{6}\mu)(2c_2 + c_3)\nabla^4 U_i \tag{49}$$

where

$$\nabla^2 g = \partial^2 g/\partial X_i \partial X_i,$$

$$\nabla^4 g = \partial^4 g/\partial X_i \partial X_i \partial X_m \partial X_m$$

and where the condition $P_{ii} = \partial U_i/\partial X_i = 0$ has been used. Equation (49) is general and does not have any connection with the yield function to this point. When $f < 0$, $\dot{P}_{ij}$ vanishes and hence increments in the displacement field caused by changes in the boundary conditions are the normal ones of linearly elastic materials. Then the particular solution of (49) are simply the residual deformations and the stresses required to equilibrate them.

When $\dot{P}_{ij} \neq 0$, the yield condition, (46) and (47) couple u_i and U_i. Suppose that the yield function is given by (16d) and it is the f_{ij} in (16d) that vanishes when $\dot{P}_{ij} \neq 0$, then

$$S_{ij} = K_{pl} t_{ijkl} + Y_{ij}$$

while

$$t_{ijkl} = \frac{c_2}{3}(P_{ij}\delta_{kl} + P_{ik}\delta_{jl} + P_{jk}\delta_{il})$$

$$+ \frac{c_3}{3}(P_{il}\delta_{jk} + \delta_{ij}P_{kl} + P_{jl}\delta_{ik}). \tag{50}$$

In the special case where $K_{kl} = k\delta_{kl}$, (50) becomes

$$S_{ij} = Y_{ij} + \frac{k}{3}\{c_2(3P_{ij} + P_{il}\delta_{jl} + P_{il} + P_{jl}) + c_3(P_{il}\delta_{jl} + P_{jk}\delta_{ik})\}.$$

In this special case, (49) becomes

$$K(4c_2 + c_3)\nabla^2 U_i + (2c_2 + c_3)\nabla^4 U_i = 0. \tag{51}$$

A similar expression can be obtained for u_i by using the yield condition and eliminating U_i. The first term is the counterpart of the elastic (or even perfectly plastic) solution. The main point is that a second term occurs in (51) which is related to the c parameters in (45). This additional term describes *deformations which are oscillatory in space* and these develop only during plastic deformations as a result of (48). They have many of the features which one expects to find in the continuum analogue of dislocations.

The condition $(\partial\phi/\partial P_{ij})\dot{P}_{ij} \geq 0$ obtained from the second law is

$$\left(S_{ij} - Y_{ij} - \frac{2c_2}{3}\alpha_{ijrr} - \frac{c_3}{3}\alpha_{ijrr}\right)\dot{P}_{ij} \geq 0. \tag{52}$$

By using (50a) and derivatives thereof (52) becomes a restriction on the permanent deformations. The result is

$$\left[k(5c_2 + 2c_3)U_{i,j} - (2c_2 + c_3)\left(1 + \frac{k}{6\mu}(5c_2 + 2c_3\right)U_{i,jrr}\right]\dot{P}_{ij} \geq 0.$$

The rate of storing energy is given by (37) and is

$$\dot{\chi} = S_{ij}(\dot{e}_{ij} - \dot{S}_{ij}/2\mu) + t_{ijkl}\dot{\alpha}_{ijkl} - q_{i,i} + \rho_0 r - \rho_0 C_D \dot{T} \tag{53}$$

or alternatively

$$\dot{\chi} = S_{ij}(\dot{e}_{ij} - \dot{S}_{ij}/2\mu) + t_{ijkl}\dot{\alpha}_{ijkl} + \rho_0 T\dot{S} + \rho_0 \frac{\partial \phi}{\partial P_{ij}} \dot{P}_{ij} - \rho_0 C_D \dot{T} \tag{53a}$$

when $\dot{W}_{\text{rev}} = S_{ij}\dot{S}_{ij}/2\mu$.

Simple shear

The results of including the higher gradients is now illustrated for the material defined by (45), with the case of simple shear in which all types of body forces and couples are absent. Simple shear implies that the deformation U_1 varies only in the thickness direction. That is

$$U_1 = K(X_2) \tag{54}$$

so that (51) becomes

$$\frac{\mathrm{d}^2 K}{\mathrm{d}X_2^2} + \omega^2 \frac{\mathrm{d}^4 K}{\mathrm{d}X_2^4} = 0 \tag{55}$$

where

$$\omega^2 = (2c_2 + c_3)/[k(4c_2 + c_3)]. \tag{56}$$

The solution of (55) is

$$K(X_2) = AX_2 + B + C\cos \omega X_2 + D \sin \omega X_2 \tag{57}$$

where the constants A, B, C and D are to be determined by the boundary conditions. The first two terms in (57) correspond to a homogeneous strain and this is all that occurs in classical plasticity where ω vanishes. The result of including the higher gradients is in the last two terms of (57). These *nonhomogeneous strain fields* develop only when the deformations are plastic.

The stresses in simple shear are given by (50) to be

$$\begin{aligned} S_{12} &= 2\mu(e_{12} - P_{12}) \\ t_{1222} &= (2c_2 + c_3)P_{12}/3 \end{aligned} \tag{58}$$

when (48) is used. The yield condition (16d) becomes

$$(S_{12} - kt_{1222} - Y_{12})(S_{12} + kt_{1222} + Y_{12}) = 0. \tag{59}$$

Suppose that it is the first term in (59) which vanishes, then the stress strain relation (58) becomes

$$S_{12} = 2\mu\left[\frac{k(2c_2 + c_3) - \frac{1}{6}\mu}{k(2c_2 + c_3)}\right] e_{12}. \tag{60}$$

By combining (58) and (59), the total strain e_{12} can be expressed in terms of the plastic strain P_{12}, and then the higher gradients are obtained by differentiation. The result is

$$\alpha_{1222} = [1 + k(2c_2 + c_3)/6\mu](\mathrm{d}^2 P_{12}/\mathrm{d}X_2^2). \tag{61}$$

The second law (52) becomes

$$[kt_{1222}-(2c_2+c_3)\alpha_{1222}/3]\dot{P}_{12} \geq 0 \tag{62}$$

by using (58) and (59). Alternatively (62) can be expressed as

$$k(2c_2+c_3)[P_{12}-(k^1/\omega^2)(\mathrm{d}^2P_{12}/\mathrm{d}X_2^2)]\dot{P}_{12} \geq 0 \tag{63}$$

where

$$k^1 = \frac{\omega^2}{k[1+k(2c_2+c_3)/6\mu]}.$$

Using the solution (57) when $D = 0$, (63) becomes

$$(A-C\omega \sin \omega X_2+k^1 C\omega \sin \omega X_2)(\dot{A}-\dot{C}\omega \sin \omega X_2) \geq 0.$$

Hence one obtains conditions that

$$\begin{aligned} A\dot{A} &\geq 0 \\ \dot{A} &\geq \omega|\dot{C}| \geq 0 \\ A &\geq \omega(1-k^1)|C| \geq 0. \end{aligned} \tag{64}$$

Similar restrictions are obtained when D does not vanish. Thus the second law imposes restriction on *processes* through limitations on boundary and initial conditions. Clearly these are the result of introducing assumption No. 5. The constitutive equations are also restricted since k^1 in (64) must be less than unity.

The stored energy in (53a) can be expressed as

$$\dot{\chi} = \frac{\mathrm{d}}{\mathrm{d}t}(t_{1222}\alpha_{1222})$$

in the case of simple shear. Alternatively it is given by

$$\dot{\chi} = \frac{(2c_2+c_3)}{3}\frac{\mathrm{d}}{\mathrm{d}t}(P_{12}\alpha_{1222}) \tag{65}$$

by virtue of (58). Hence when the strains are homogeneous there is no increase in the stored energy. The result is consistent with considerable experimental data on stored energy as determined by metallurgists. The energy that is stored is determined by the same parameters (c_2, c_3 and k) that define the work hardening coefficient in (60) and (65).

The energy equation (36) reduces to the heat conduction one with a source term. It is

$$-q_{\alpha,\alpha}+\rho_0 r = \rho_0 C_D \dot{T}-[S_{12}-(2c_2+c_3)\alpha_{1222}/3]\dot{P}_{12}. \tag{66}$$

The last term is never negative by virtue of (59) and (62). It is possible to show that the source term in (66) is

$$S_{12}\dot{P}_{12}+t_{1222}\dot{\alpha}_{1222}-\dot{\chi}$$

when $\dot{\chi}$ is given by (65).

7. COMMENTS

Modern research in plasticity has many goals, some of which may not be consistent with one another. Our purpose here is to develop a continuum theory which permits behavior on a small scale (microscopic) to influence large scale (macroscopic) response in a logically consistent manner. Experimental evidence indicates that the deformations are not homogeneous on the small scale when they are at the macroscopic level. We have accomplished the purpose of having the response at the two scales interact and at the same time found the nonhomogeneous deformations on the small scale as a result. We have done this by using higher displacement gradients, which in turn was motivated by the atomistic concept of work hardening being due to dislocation interaction. In turn these imply *long range* forces which are not of the contact (nearest neighbor) type. To illustrate the entire picture we had to consider a very special free energy and therefore the results are probably limited in their application. However considerable insight can be obtained by this particular model even if the details are not adequate.

Gradient theories are not the only way of introducing the interaction between the two levels of observing nature. Internal variables have recently [3, 21] been used to reproduce phenomenological results. The internal variable approach permits one to utilize results known to metallurgists and is certainly more versatile than gradient theories. However the internal variable method does not contain the local approximation to the deformation field while the gradient type theories do. A major problem in the gradient theories is the choice of those gradients which are to be used. Moreover the solution of boundary valued problems in the nonlinear range can be expected to be very difficult.

The continuous distribution of dislocations [14] can be used to calculate the increase in stored energy when one assumes (or measures) the distribution of dislocations. The gradient theory described above requires many more formal assumptions than the continuous distribution of dislocations but at the same time it can be used during the process of plastic deformation and predicts how energy is stored.

It is well known that slip takes place on a few discrete planes between which there is large amounts of material that does not yield. Usually the mathematical theory of plasticity ignores this and assumes homogeneous plastic flow in a tensile test specimen. The present theory predicts regions of greater (and lesser) residual deformations, but not the discrete planes such as are actually observed.

8. CONCLUSIONS

A continuum model of plasticity which permits small scale response to interact and influence large scale behavior has been developed. Higher strain gradients have been used and these in turn lead to nonhomogeneous strain fields in simple shear.

The main value of this model appears to be the conceptual one because it permits one to better understand the fundamental role of small scale response on observed behavior.

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Абстракт—В качестве модели для пластических деформаций, предполагается теория, заключающая градиенты деформации первого и второго порядка. Даются, также, эвристические аргументы для учета градиентов. Главным доказательством является развитие логической схемы, в которой поведение в малом масштабе может взаииодействовать с воздействием в большом масштабе. Градиенты приводят к деформациям, которые содержат колебателвные составляющие при добавке к таким же самым, полученным без них. Для частного материала, зти деформации возникшие вследсвие осуцлаций, возникают только во время пластических деформаций. Они обладают в большинстве свойствами, которых можно ожидатъ при сплошной аналогии дислокаций. В противопопоножности с теорией сплошного распределения дислокаций, предлагаемая теория применяется к всему процессу пластических Деформаций.